## CHAPTER 9 INTRODUCING NONCOMMUTATIVE ALGEBRA

We will discuss noncommutative rings and their modules, concentrating on two fundamental results, the Wedderburn structure theorem and Maschke's theorem. Further insight into the structure of rings will be provided by the Jacobson radical.

### 9.1 Semisimple Modules

A vector space is the direct sum of one-dimensional subspaces (each subspace consists of scalar multiples of a basis vector). A one-dimensional space is simple in the sense that it does not have a nontrivial proper subspace. Thus any vector space is a direct sum of simple subspaces. We examine those modules which behave in a similar manner.
9.1.1 Definition An $R$-module $M$ is simple if $M \neq 0$ and the only submodules of $M$ are 0 and $M$.
9.1.2 Theorem Let $M$ be a nonzero $R$-module. The following conditions are equivalent, and a module satisfying them is said to be semisimple or completely reducible.
(a) $M$ is a sum of simple modules;
(b) $M$ is a direct sum of simple modules;
(c) If $N$ is a submodule of $M$, then $N$ is a direct summand of $M$, that is, there is a submodule $N^{\prime}$ of $M$ such that $M=N \oplus N^{\prime}$.

Proof.
(a) implies (b). Let $M$ be the sum of simple modules $M_{i}, i \in I$. If $J \subseteq I$, denote $\sum_{j \in J} M_{j}$ by $M(J)$. By Zorn's lemma, there is a maximal subset $J$ of $I$ such that the sum defining $N=M(J)$ is direct. We will show that $M=N$. First assume that $i \notin J$. Then $N \cap M_{i}$ is a submodule of the simple module $M_{i}$, so it must be either 0 or $M_{i}$. If $N \cap M_{i}=0$, then $M(J \cup\{i\})$ is direct, contradicting maximality of $J$. Thus $N \cap M_{i}=M_{i}$, so $M_{i} \subseteq N$. But if $i \in J$, then $M_{i} \subseteq N$ by definition of $N$. Therefore $M_{i} \subseteq N$ for all $i$, and since $M$ is the sum of all the $M_{i}$, we have $M=N$.
(b) implies (c). This is essentially the same as (a) implies (b). Let $N$ be a submodule of $M$, where $M$ is the direct sum of simple modules $M_{i}, i \in I$. Let $J$ be a maximal subset of $I$ such that the sum $N+M(J)$ is direct. If $i \notin J$ then exactly as before, $M_{i} \cap(N \oplus M(J))=M_{i}$, so $M_{i} \subseteq N \oplus M(J)$. This holds for $i \in J$ as well, by definition of $M(J)$. It follows that $M=N \oplus M(J)$. [Notice that the complementary submodule $N^{\prime}$ can be taken as a direct sum of some of the original $M_{i}$.]
(c) implies (a). First we make several observations.
(1) If $M$ satisfies (c), so does every submodule $N$.
[Let $N \leq M$, so that $M=N \oplus N^{\prime}$. If $V$ is a submodule of $N$, hence of $M$, we have $M=V \oplus W$. If $x \in N$, then $x=v+w, v \in V, w \in W$, so $w=x-v \in N$ (using $V \leq N)$. But $v$ also belongs to $N$, and consequently $N=(N \cap V) \oplus(N \cap W)=V \oplus(N \cap W)$.]
(2) If $D=A \oplus B \oplus C$, then $A=(A+B) \cap(A+C)$.
[If $a+b=a^{\prime}+c$, where $a, a^{\prime} \in A, b \in B, c \in C$, then $a^{\prime}-a=b-c$, and since $D$ is a direct sum, we have $b=c=0$ and $a=a^{\prime}$. Thus $a+b \in A$.]
(3) If $N$ is a nonzero submodule of $M$, then $N$ contains a simple submodule.
[Choose a nonzero $x \in N$. By Zorn's lemma, there is a maximal submodule $V$ of $N$ such that $x \notin V$. By (1) we can write $N=V \oplus V^{\prime}$, and $V^{\prime} \neq 0$ by choice of $x$ and $V$. If $V^{\prime}$ is simple, we are finished, so assume the contrary. Then $V^{\prime}$ contains a nontrivial proper submodule $V_{1}$, so by (1) we have $V^{\prime}=V_{1} \oplus V_{2}$ with the $V_{j}$ nonzero. By (2), $V=\left(V+V_{1}\right) \cap\left(V+V_{2}\right)$. Since $x \notin V$, either $x \notin V+V_{1}$ or $x \notin V+V_{2}$, which contradicts the maximality of $V$.]

To prove that (c) implies (a), let $N$ be the sum of all simple submodules of $M$. By (c) we can write $M=N \oplus N^{\prime}$. If $N^{\prime} \neq 0$, then by (3), $N^{\prime}$ contains a simple submodule $V$. But
then $V \leq N$ by definition of $N$. Thus $V \leq N \cap N^{\prime}=0$, a contradiction. Therefore $N^{\prime}=0$ and $M=N$.
9.1.3 Proposition Nonzero submodules and quotient modules of a semisimple module are semisimple.

Proof. The submodule case follows from (1) of the proof of (9.1.2). Let $N \leq M$, where $M=\sum_{i} M_{i}$ with the $M_{i}$ simple. Applying the canonical map from $M$ to $M / N$, we have

$$
M / N=\sum_{i}\left(M_{i}+N\right) / N
$$

This key idea has come up before; see the proofs of (1.4.4) and (4.2.3). By the second isomorphism theorem, $\left(M_{i}+N\right) / N$ is isomorphic to a quotient of the simple module $M_{i}$. But a quotient of $M_{i}$ is isomorphic to $M_{i}$ or to zero, and it follows that $M / N$ is a sum of simple modules. By (a) of (9.1.2), $M / N$ is semisimple.

## Problems For Section 9.1

1. Regard a ring $R$ as an $R$-module. Show that $R$ is simple if and only if $R$ is a division ring.
2. Let $M$ be an $R$-module, with $x$ a nonzero element of $M$. Define the $R$-module homomorphism $f: R \rightarrow R x$ by $f(r)=r x$. Show that the kernel $I$ of $f$ is a proper ideal of $R$, and $R / I$ is isomorphic to $R x$.
3. If $M$ is a nonzero $R$-module, show that $M$ is simple if and only if $M \cong R / I$ for some maximal left ideal $I$.
4. If $M$ is a nonzero $R$-module, show that $M$ is simple if and only if $M$ is cyclic (that is, $M$ can be generated by a single element) and every nonzero element of $M$ is a generator.
5. What do simple $\mathbb{Z}$-modules look like?
6. If $F$ is a field, what do simple $F[X]$-modules look like?
7. Let $V$ be an $n$-dimensional vector space over a field $k$. (Take $n \geq 1$ so that $V \neq 0$.) If $f$ is an endomorphism (that is, a linear transformation) of $V$ and $x \in V$, define $f x=f(x)$. This makes $V$ into a module over the endomorphism ring $\operatorname{End}_{k}(V)$. Show that the module is simple.
8. Show that a nonzero module $M$ is semisimple if and only if every short exact sequence $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$ splits.

### 9.2 Two Key Theorems

If $M$ is a simple $R$-module, there are strong restrictions on a homomorphism either into or out of $M$. A homomorphism from one simple $R$-module to another is very severely restricted, as Schur's lemma reveals. This very useful result will be important in the proof of Wedderburn's structure theorem. Another result that will be needed is a theorem of Jacobson that gives some conditions under which a module homomorphism $f$ amounts to multiplication by a fixed element of a ring, at least on part of the domain of $f$.

### 9.2.1 Schur's Lemma

(a) If $f \in \operatorname{Hom}_{R}(M, N)$ where $M$ and $N$ are simple $R$-modules, then $f$ is either identically 0 or an isomorphism.
(b) If $M$ is a simple $R$-module, then $\operatorname{End}_{R}(M)$ is a division ring.

Proof.
(a) The kernel of $f$ is either 0 or $M$, and the image of $f$ is either 0 or $N$. If $f$ is not the zero map, then the kernel is 0 and the image is $N$, so $f$ is an isomorphism.
(b) Let $f \in \operatorname{End}_{R}(M), f$ not identically 0 . By (a), $f$ is an isomorphism, and therefore is invertible in the endomorphism ring of $M$.

The next result prepares for Jacobson's theorem.
9.2.2 Lemma Let $M$ be a semisimple $R$-module, and let $A$ be the endomorphism ring $\operatorname{End}_{R}(M)$. [Note that $M$ is an $A$-module; if $g \in A$ we take $g \bullet x=g(x), x \in M$.] If $m \in M$ and $f \in \operatorname{End}_{A}(M)$, then there exists $r \in R$ such that $f(m)=r m$.

Before proving the lemma, let's look more carefully at $\operatorname{End}_{A}(M)$. Suppose that $f \in$ $\operatorname{End}_{A}(M)$ and $x \in M$. If $g \in A$ then $f(g(x))=g(f(x))$. Thus $\operatorname{End}_{A}(M)$ consists of those abelian group endomorphisms of $M$ that commute with everything in $\operatorname{End}_{R}(M)$. In turn, by the requirement that $f(r x)=r f(x), \operatorname{End}_{R}(M)$ consists of those abelian group endomorphisms of $M$ that commute with $R$, more precisely with multiplication by $r$, for each $r \in R$. For this reason, $\operatorname{End}_{A}(M)$ is sometimes called the double centralizer of $R$.

We also observe that the map taking $r \in R$ to multiplication by $r$ is a ring homomorphism of $R$ into $\operatorname{End}_{A}(M)$. [Again use $r f(x)=f(r x)$.] Jacobson's theorem will imply that given any $f$ in $\operatorname{End}_{A}(M)$ and any finite set $S \subseteq M$, some $g$ in the image of this ring homomorphism will agree with $f$ on $S$. Thus in (9.2.2), we can replace the single element $m$ by an arbitrary finite subset of $M$.

Proof. By (9.1.2) part (c), we can express $M$ as a direct sum $R m \oplus N$. Now if we have a direct sum $U=V \oplus W$ and $u=v+w, v \in V, w \in W$, there is a natural projection of $U$ on $V$, namely $u \rightarrow v$. In the present case, let $\pi$ be the natural projection of $M$ on $R m$. Then $\pi \in A$ and $f(m)=f(\pi m)=\pi f(m) \in R m$. The result follows.

Before proving Jacobson's theorem, we review some ideas that were introduced in the exercises in Section 4.4.
9.2.3 Comments To specify an $R$-module homomorphism $\psi$ from a direct sum $V^{*}=$ $\oplus_{j=1}^{n} V_{j}$ to a direct sum $W^{*}=\oplus_{i=1}^{m} W_{i}$, we must give, for every $i$ and $j$, the $i^{\text {th }}$ component of the image of $v_{j} \in V_{j}$. Thus the homomorphism is described by a matrix $\left[\psi_{i j}\right]$, where $\psi_{i j}$ is a homomorphism from $V_{j}$ to $W_{i}$. The $i^{\text {th }}$ component of $\psi\left(v_{j}\right)$ is $\psi_{i j}\left(v_{j}\right)$, so the $i^{t h}$ component of $\psi\left(v_{1}+\cdots+v_{n}\right)$ is $\sum_{j=1}^{n} \psi_{i j}\left(v_{j}\right)$. Consequently,

$$
\psi\left(v_{1}+\cdots+v_{n}\right)=\left[\psi_{i j}\right]\left[\begin{array}{c}
v_{1}  \tag{1}\\
\vdots \\
v_{n}
\end{array}\right] .
$$

This gives an abelian group isomorphism between $\operatorname{Hom}_{R}\left(V^{*}, W^{*}\right)$ and $\left[\operatorname{Hom}_{R}\left(V_{j}, W_{i}\right)\right]$, the collection of all $m$ by $n$ matrices whose $i j$ entry is an $R$-module homomorphism from $V_{j}$ to $W_{i}$. If we take $m=n$ and $V_{i}=W_{j}=V$ for all $i$ and $j$, then $V^{*}=W^{*}=V^{n}$, the direct sum of $n$ copies of $V$. Then the abelian group isomorphism given by (1) becomes

$$
\begin{equation*}
\operatorname{End}_{R}\left(V^{n}\right) \cong M_{n}\left(\operatorname{End}_{R}(V)\right) \tag{2}
\end{equation*}
$$

the collection of all $n$ by $n$ matrices whose entries are $R$-endomorphisms of $V$. Since composition of endomorphisms corresponds to multiplication of matrices, (2) gives a ring isomorphism as well.
9.2.4 Theorem (Jacobson) Let $M$ be a semisimple $R$-module, and let $A$ be the endomorphism ring $\operatorname{End}_{R}(M)$. If $f \in \operatorname{End}_{A}(M)$ and $m_{1}, \ldots, m_{n} \in M$, then there exists $r \in R$ such that $f\left(m_{i}\right)=r m_{i}$ for all $i=1, \ldots, n$.
Proof. $f$ induces an endomorphism $f^{(n)}$ of $M^{n}$, the direct sum of $n$ copies of $M$, via

$$
f^{(n)}\left(m_{1}+\cdots+m_{n}\right)=f\left(m_{1}\right)+\cdots+f\left(m_{n}\right)
$$

where $f\left(m_{i}\right)$ belongs to the $i^{t h}$ copy of $M$. Thus the matrix that represents $f^{(n)}$ is the scalar matrix $f I$, where $I$ is an $n$ by $n$ identity matrix. If $B=\operatorname{End}_{R}\left(M^{n}\right)$, then since a
scalar matrix commutes with everything, $f^{(n)} \in \operatorname{End}_{B}\left(M^{n}\right)$. If $m_{1}, \ldots, m_{n} \in M$, then by (9.2.2), there exists $r \in R$ such that $f^{(n)}\left(m_{1}+\cdots m_{n}\right)=r\left(m_{1}+\cdots m_{n}\right)$. [Note that since $M$ is semisimple, so is $M^{n}$.] This is equivalent to $f\left(m_{i}\right)=r m_{i}$ for all $i$.

Before giving a corollary, we must mention that the standard results that every vector space over a field has a basis, and any two bases have the same cardinality, carry over if the field is replaced by a division ring. Also recall that a module is said to be faithful if its annihilator is 0 .
9.2.5 Corollary Let $M$ be a faithful, simple $R$-module, and let $D=\operatorname{End}_{R}(M)$, a division ring by $(9.2 .1(\mathrm{~b}))$. If $M$ is a finite-dimensional vector space over $D$, then $\operatorname{End}_{D}(M) \cong R$, a ring isomorphism.

Proof. Let $\left\{x_{1}, \ldots, x_{n}\right\}$ be a basis for $M$ over $D$. By (9.2.4), if $f \in \operatorname{End}_{D}(M)$, there exists $r \in R$ such that $f\left(x_{i}\right)=r x_{i}$ for all $i=1, \ldots, n$. Since the $x_{i}$ form a basis, we have $f(x)=r x$ for every $x \in M$. Thus the map $h$ from $R$ to $\operatorname{End}_{D}(M)$ given by $r \rightarrow g_{r}=$ multiplication by $r$ is surjective. If $r x=0$ for all $x \in M$, then since $M$ is faithful, we have $r=0$ and $h$ is injective. Since $h(r s)=g_{r} \circ g_{s}=h(r) h(s), h$ is a ring isomorphism.

## Problems For Section 9.2

1. Criticize the following argument. Let $M$ be a simple $R$-module, and let $A=\operatorname{End}_{R}(M)$. "Obviously" $M$ is also a simple $A$-module. For any additive subgroup $N$ of $M$ that is closed under the application of all $R$-endomorphisms of $M$ is, in particular, closed under multiplication by an element $r \in R$. Thus $N$ is an $R$-submodule of $M$, hence is 0 or $M$.
2. Let $M$ be a nonzero cyclic module. Show that $M$ is simple if and only if ann $M$, the annihilator of $M$, is a maximal left ideal.
3. In Problem 2, show that the hypothesis that $M$ is cyclic is essential.
4. Let $V=F^{n}$ be the $n$-dimensional vector space of all $n$-tuples with components in the field $F$. If $T$ is a linear transformation on $V$, then $V$ becomes an $F[X]$-module via $f(X) v=f(T) v$. For example, if $n=2, T(a, b)=(0, a)$, and $f(X)=a_{0}+a_{1} X+\cdots+a_{n} X^{n}$, then

$$
\begin{aligned}
f(X)(1,0) & =a_{0}(1,0)+a_{1} T(1,0)+a_{2} T^{2}(1,0)+\cdots+a_{n} T^{n}(1,0) \\
& =\left(a_{0}, 0\right)+\left(0, a_{1}\right) \\
& =\left(a_{0}, a_{1}\right)
\end{aligned}
$$

Show that in this case, $V$ is cyclic but not simple.
5. Suppose that $M$ is a finite-dimensional vector space over an algebraically closed field $F$, and in addition $M$ is a module over a ring $R$ containing $F$ as a subring. If $M$ is a simple $R$-module and $f$ is an $R$-module homomorphism, in particular an $F$-linear transformation, on $M$, show that $f$ is multiplication by some fixed scalar $\lambda \in F$. This result is frequently given as a third part of Schur's lemma.
6. Let $I$ be a left ideal of the $\operatorname{ring} R$, so that $R / I$ is an $R$-module but not necessarily a ring. Criticize the following statement: "Obviously", $I$ annihilates $R / I$.

### 9.3 Simple and Semisimple Rings

9.3.1 Definitions and Comments Since a ring is a module over itself, it is natural to call a ring $R$ semisimple if it is semisimple as an $R$-module. Our aim is to determine, if possible, how semisimple rings are assembled from simpler components. A plausible idea is that the components are rings that are simple as modules over themselves. But this turns out to be too restrictive, since the components would have to be division rings (Section 9.1, Problem 1).

When we refer to a simple left ideal $I$ of $R$, we will always mean that $I$ is simple as a left $R$-module. We say that the ring $R$ is simple if $R$ is semisimple and all simple left ideals of $R$ are isomorphic. [The definition of simple ring varies in the literature. An advantage of
our choice (also favored by Lang and Bourbaki) is that we avoid an awkward situation in which a ring is simple but not semisimple.] Our goal is to show that the building blocks for semisimple rings are rings of matrices over a field, or more generally, over a division ring.

The next two results give some properties of modules over semisimple rings.
9.3.2 Proposition If $R$ is a semisimple ring, then every nonzero $R$-module $M$ is semisimple.

Proof. By (4.3.6), $M$ is a quotient of a free $R$-module $F$. Since $F$ is a direct sum of copies of $R$ (see (4.3.4)), and $R$ is semisimple by hypothesis, it follows from (9.1.2) that $F$ is semisimple. By (9.1.3), $M$ is semisimple.
9.3.3 Proposition Let $I$ be a simple left ideal in the semisimple ring $R$, and let $M$ be a simple $R$-module. Denote by $I M$ the $R$-submodule of $M$ consisting of all finite linear combinations $\sum_{i} r_{i} x_{i}, r_{i} \in I, x_{i} \in M$. Then either $I M=M$ and $I$ is isomorphic to $M$, or $I M=0$.

Proof. If $I M \neq 0$, then since $M$ is simple, $I M=M$. Thus for some $x \in M$ we have $I x \neq 0$, and again by simplicity of $M$, we have $I x=M$. Map $I$ onto $M$ by $r \rightarrow r x$, and note that the kernel cannot be $I$ because $I x \neq 0$. Since $I$ is simple, the kernel must be 0 , so $I \cong M$.

### 9.3.4 Beginning the Decomposition

Let $R$ be a semisimple ring. We regard two simple left ideals of $R$ as equivalent if they are isomorphic (as $R$-modules), and we choose a representative $I_{i}, i \in T$ from each equivalence class. We define the basic building blocks of $R$ as

$$
B_{i}=\text { the sum of all left ideals of } R \text { that are isomorphic to } I_{i} \text {. }
$$

We have a long list of properties of the $B_{i}$ to establish, and for the sake of economy we will just number the statements and omit the words "Lemma" and "Proof" in each case. We will also omit the end of proof symbol, except at the very end.
9.3.5 If $i \neq j$, then $B_{i} B_{j}=0$. [The product of two left ideals is defined exactly as in (9.3.3).]

Apply (9.3.3) with $I$ replaced by $B_{i}$ and $M$ by $B_{j}$.
9.3.6 $R=\sum_{i \in T} B_{i}$

If $r \in R$, then $(r)$ is a left ideal, which by (9.1.2) and (9.1.3) (or (9.3.2)) is a sum of simple left ideats.
9.3.7 Each $B_{i}$ is a two-sided ideal.

Using (9.3.5) and (9.3.6) we have

$$
B_{i} \subseteq B_{i} R=B_{i} \sum_{j} B_{j}=B_{i} B_{i} \subseteq R B_{i} \subseteq B_{i}
$$

Thus $R B_{i}=B_{i} R=B_{i}$.
9.3.8 $R$ has only finitely many isomorphism classes of simple left ideals $I_{1}, \ldots, I_{t}$.

By (9.3.6), we can write the identity 1 of $R$ as a finite sum of elements $e_{i} \in B_{i}, i \in T$. Adjusting the notation if necessary, let $1=\sum_{i=1}^{t} e_{i}$. If $r \in B_{j}$ where $j \notin\{1, \ldots, t\}$, then by (9.3.5), $r e_{i}=0$ for all $i=1, \ldots, t$, so $r=r 1=0$. Thus $B_{j}=0$ for $j \notin\{1, \ldots, t\}$.
9.3.9 $R=\oplus_{i=1}^{t} B_{i}$. Thus 1 has a unique representation as $\sum_{i=1}^{t} e_{i}$, with $e_{i} \in B_{i}$.

By (9.3.6) and (9.3.8), $R$ is the sum of the $B_{i}$. If $b_{1}+\cdots+b_{t}=0$, with $b_{i} \in B_{i}$, then

$$
0=e_{i}\left(b_{1}+\cdots+b_{t}\right)=e_{i} b_{1}+\cdots e_{i} b_{t}=e_{i} b_{i}=\left(e_{1}+\cdots+e_{t}\right) b_{i}=1 b_{i}=b_{i}
$$

Therefore the sum is direct.
9.3.10 If $b_{i} \in B_{i}$, then $e_{i} b_{i}=b_{i}=b_{i} e_{i}$. Thus $e_{i}$ is the identity on $B_{i}$ and $B_{i}=R e_{i}=e_{i} R$.

The first assertion follows from the computation in (9.3.9), along with a similar computation with $e_{i}$ multiplying on the right instead of the left. Now $B_{i} \subseteq R e_{i}$ because $b_{i}=b_{i} e_{i}$, and $R e_{i} \subseteq B_{i}$ by (9.3.7) and the fact that $e_{i} \in B_{i}$. The proof that $B_{i}=e_{i} R$ is similar.
9.3.11 Each $B_{i}$ is a simple ring.

By the computation in (9.3.7), along with (9.3.10), $B_{i}$ is a ring (with identity $e_{i}$ ). Let $J$ be a simple left ideal of $B_{i}$. By (9.3.5) and (9.3.6), $R J=B_{i} J=J$, so $J$ is a left ideal of $R$, necessarily simple. Thus $J$ is isomorphic to some $I_{j}$, and we must have $j=i$. [Otherwise, $J$ would appear in the sums defining both $B_{i}$ and $B_{j}$, contradicting (9.3.9).] Therefore $B_{i}$ has only one isomorphism class of simple left ideals. Now $B_{i}$ is a sum of simple left ideals of $R$, and a subset of $B_{i}$ that is a left ideal of $R$ must be a left ideal of $B_{i}$. Consequently, $B_{i}$ is semisimple and the result follows.
9.3.12 If $M$ is a simple $R$-module, then $M$ is isomorphic to some $I_{i}$. Thus there are only finitely many isomorphism classes of simple $R$-modules. In particular, if $R$ is a simple ring, then all simple $R$-modules are isomorphic.
By (9.3.9),

$$
R=\sum_{i=1}^{t} B_{i}=\sum_{i=1}^{t} \sum\left\{J: J \cong I_{i}\right\}
$$

where the $J$ are simple left ideals. Therefore

$$
M=R M=\sum_{i=1}^{t} B_{i} M=\sum_{i=1}^{t} \sum\left\{J M: J \cong I_{i}\right\}
$$

By (9.3.3), $J M=0$ or $J \cong M$. The former cannot hold for all $J$, since $M \neq 0$. Thus $M \cong I_{i}$ for some $i$. If $R$ is a simple ring, then there is only one $i$, and the result follows.
9.3.13 Let $M$ be a nonzero $R$-module, so that $M$ is semisimple by (9.3.2). Define $M_{i}$ as the sum of all simple submodules of $M$ that are isomorphic to $I_{i}$, so that by (9.3.12), $M=\sum_{i=1}^{t} M_{i}$. Then

$$
M=\bigoplus_{i=1}^{t} B_{i} M \text { and } B_{i} M=e_{i} M=M_{i}, i=1, \ldots, t
$$

By definition of $B_{i}$,

$$
B_{i} M_{j}=\sum\left\{J M_{j}: J \cong I_{i}\right\}
$$

where the $J$ 's are simple left ideals. If $N$ is any simple module involved in the definition of $M_{j}$, then $J N$ is 0 or $N$, and by (9.3.3), $J N=N$ implies that $N \cong J \cong I_{i}$. But all such $N$ are isomorphic to $I_{j}$, and therefore $B_{i} M_{j}=0, i \neq j$. Thus

$$
M_{i}=R M_{i}=\sum_{j} B_{j} M_{i}=B_{i} M_{i}
$$

and

$$
B_{i} M=\sum_{j} B_{i} M_{j}=B_{i} M_{i}
$$

Consequently, $M_{i}=B_{i} M=e_{i} R M=e_{i} M$ (using (9.3.10)), and all that remains is to show that the sum of the $M_{i}$ is direct. Let $x_{1}+\cdots+x_{t}=0, x_{i} \in M_{i}$. Then

$$
0=e_{i}\left(x_{1}+\cdots+x_{t}\right)=e_{i} x_{i}
$$

since $e_{i} x_{j} \in B_{i} M_{j}=0$ for $i \neq j$. Finally, by (9.3.9),

$$
e_{i} x_{i}=\left(e_{1}+\cdots+e_{t}\right) x_{i}=x_{i} .
$$

9.3.14 A semisimple ring $R$ is ring-isomorphic to a direct product of simple rings.

This follows from (9.3.9) and (9.3.5). For if $a_{i}, b_{i} \in B_{i}$, then

$$
\left(a_{1}+\cdots+a_{t}\right)\left(b_{1}+\cdots+b_{t}\right)=a_{1} b_{1}+\cdots+a_{t} b_{t}
$$

## Problems For Section 9.3

In Problems 1 and 2, let $M$ be a semisimple module, so that $M$ is the direct sum of simple modules $M_{i}, i \in I$. We are going to show that $M$ is a finite direct sum of simple modules if and only if $M$ is finitely generated.

1. Suppose that $x_{1}, \ldots, x_{n}$ generate $M$. It will follow that $M$ is the direct sum of finitely many of the $M_{i}$. How would you determine which $M_{i}$ 's are involved?
2. Conversely, assume that $M$ is a finite sum of simple modules. Show that $M$ is finitely generated.
3. A left ideal $I$ is said to be minimal if $I \neq 0$ and $I$ has no proper subideal except 0 . Show that the ring $R$ is semisimple if and only if $R$ is a direct sum of minimal left ideals.
4. Is $\mathbb{Z}$ semisimple?
5. Is $\mathbb{Z}_{n}$ semisimple?
6. Suppose that $R$ is a ring with the property that every nonzero $R$-module is semisimple. Show that every $R$-module $M$ is projective, that is, every exact sequence $0 \rightarrow A \rightarrow B \rightarrow$ $M \rightarrow 0$ splits. Moreover, $M$ is injective, that is, every exact sequence $0 \rightarrow M \rightarrow A \rightarrow B \rightarrow 0$ splits. [Projective and injective modules will be studied in Chapter 10.]
7. For any ring $R$, show that the following conditions are equivalent.
(a) $R$ is semisimple;
(b) Every nonzero $R$-module is semisimple;
(c) Every $R$-module is projective;
(d) Every $R$-module is injective.

### 9.4 Further Properties of Simple Rings, Matrix Rings, and Endomorphisms

To reach the Wedderburn structure theorem, we must look at simple rings in more detail, and supplement what we already know about matrix rings and rings of endomorphisms.
9.4.1 Lemma Let $R$ be any ring, regarded as a left module over itself. If $h: R \rightarrow M$ is an $R$ module homomorpbism, then for some $x \in M$ we have $h(r)=r x$ for every $r \in R$. Moreover, we may choose $x=h(1)$, and the map $h \rightarrow h(1)$ is an isomorphism of $\operatorname{Hom}_{R}(R, M)$ and $M$. This applies in particular when $M=R$, in which case $h \in \operatorname{End}_{R}(R)$.

Proof. The point is that $h$ is determined by what it does to the identity. Thus

$$
h(r)=h(r 1)=r h(1)
$$

so we may take $x=h(1)$. If $s \in R$ and $h \in \operatorname{Hom}_{R}(R, M)$, we take $(s h)(r)=h(r s)=r s x$. This makes $\operatorname{Hom}_{R}(R, M)$ into a left $R$-module isomorphic to $M$. (For further discussion of this idea, see the exercises in Section 10.7.)

Notice that although all modules are left $R$-modules, $h$ is given by multiplication on the right by $x$.
9.4.2 Corollary Let $I$ and $J$ be simple left ideals of the simple ring $R$. Then for some $x \in R$ we have $J=I x$.

Proof. By the definition of a simple ring (see (9.3.1)), $R$ is semisimple, so by (9.1.2), $R=I \oplus L$ for some left ideal $L$. Again by the definition of a simple ring, $I$ and $J$ are isomorphic (as $R$-modules). If $\tau: I \rightarrow J$ is an isomorphism and $\pi$ is the natural projection of $R$ on $I$, then $\tau \pi \in \operatorname{End}_{R}(R)$, so by (9.4.1), there exists $x \in R$ such that $\tau \pi(r)=r x$ for every $r \in R$. Allow $r$ to range over $I$ to conclude that $J=I x$.

A semisimple ring can be expressed as a direct sum of simple left ideals, by (9.1.2). If the ring is simple, only finitely many simple left ideals are needed.
9.4.3 Lemma A simple ring $R$ is a finite direct sum of simple left ideals.

Proof. Let $R=\oplus_{j} I_{j}$ where the $I_{j}$ are simple left ideals. Changing notation if necessary, we have $1=y_{1}+\cdots+y_{m}$ with $y_{j} \in I_{j}, j=1, \ldots, m$. If $x \in R$, then

$$
x=x 1=\sum_{j=1}^{m} x y_{j} \in \sum_{j=1}^{m} I_{j} .
$$

Therefore $R$ is a finite sum of the $I_{j}$, and the sum is direct because the original decomposition of $R$ is direct.
9.4.4 Corollary If $I$ is a simple left ideal of the simple ring $R$, then $I R=R$.

Proof. If $J$ is any simple left ideal of $R$, then by (9.4.2), $J \subseteq I R$. By (9.4.3), $R$ is a finite (direct) sum of simple left ideals, so $R \subseteq I R$. The reverse inclusion always holds, and the result follows.

We now have some insight into the structure of simple rings.
9.4.5 Proposition If $R$ is a simple ring, then the only two-sided ideals of $R$ are 0 and $R$.

Proof. Let $J$ be a nonzero 2 -sided ideal of $R$. By (9.1.3), $J$ is a semisimple left $R$-module, so by (9.1.2), $J$ is a sum of simple left ideals of $J$, hence of $R$. In particular, $J$ contains a simple left ideal $I$. Since $J$ is a right ideal, it follows that $J=J R$. Using (9.4.4), we have

$$
J=J R \supseteq I R=R
$$

so $J=R$.
In the literature, a simple ring is often defined as a ring $R$ whose only two-sided ideals are 0 and $R$, but then extra hypotheses must be added to force $R$ to be semisimple. See the exercises for further discussion.
9.4.6 Corollary Let $I$ be a simple left ideal of the simple ring $R$, and let $M$ be a simple $R$-module. Then $I M=M$ and $M$ is faithful.

Proof. The first assertion follows from a computation that uses associativity of scalar multiplication in a module, along with (9.4.4):

$$
\begin{equation*}
M=R M=(I R) M=I(R M)=I M . \tag{1}
\end{equation*}
$$

Now let $b$ belong to the annihilator of $M$, so that $b M=0$. We must show that $b=0$. By a computation similar to (1) (using in addition the associativity of ring multiplication),

$$
\begin{equation*}
R b R M=R b M=R 0=0 . \tag{2}
\end{equation*}
$$

But $R b R$ is a two-sided ideal of $R$ (see (2.2.7)), so by (9.4.5), $R b R=0$ or $R$. In the latter case, $M=R M=R b R M=0$ by (2), contradicting the assumption that $M$ is simple. Therefore $R b R=0$, in particular, $b=1 b 1=0$.

We are now ready to show that a simple ring is isomorphic to a ring of matrices. Let $R$ be a simple ring, and $V$ a simple $R$-module. [ $V$ exists because $R$ is a sum of simple left ideals, and $V$ is unique up to isomorphism by (9.3.12).] Let $D=\operatorname{End}_{R}(V)$, a division ring by Schur's lemma (9.2.1(b)). Then (see (9.2.2)), $V$ is a $D$-module, in other words, a vector space over $D . V$ is a faithful $R$-module by (9.4.6), and if we can prove that $V$ is finitedimensional as a vector space over $D$, then by (9.2.5), $R$ is ring-isomorphic to $\operatorname{End}_{D}(V)$. If $n$ is the dimension of $V$ over $D$, then by (4.4.1), $\operatorname{End}_{D}(V) \cong M_{n}\left(D^{o}\right)$, the ring of $n$ by $n$ matrices with entries in the opposite ring $D^{\circ}$.
9.4.7 Theorem Let $R$ be a simple ring, $V$ a simple $R$-module, and $D$ the endomorphism ring $\operatorname{End}_{R}(V)$. Then $V$ is a finite-dimensional vector space over $D$. If the dimension of this vector space is $n$, then (by the above discussion),

$$
R \cong \operatorname{End}_{D}(V) \cong M_{n}\left(D^{o}\right)
$$

Proof. Assume that we have infinitely many linearly independent elements $x_{1}, x_{2}, \ldots$ Let $I_{m}$ be the left ideal $\left\{r \in R: r x_{i}=0\right.$ for all $\left.i=1, \ldots, m\right\}$. Then the $I_{m}$ decrease as $m$ increases, in fact they decrease strictly. [Given any $m$, let $f$ be a $D$-linear transformation on $V$ such that $f\left(x_{i}\right)=0$ for $1 \leq i \leq m$ and $f\left(x_{m+1}\right) \neq 0$. By Jacobson's theorem (9.2.4), there exists $r \in R$ such that $f\left(x_{i}\right)=r x_{i}, i=1, \ldots, m+1$. But then $r x_{1}=\cdots=r x_{m}=0, r x_{m+1} \neq 0$, so $r \in I_{m} \backslash I_{m+1}$.] Write $I_{m}=J_{m} \oplus I_{m+1}$, as in (9.1.2) part (c). [Recall from (9.1.3) that since $R$ is semisimple, so are all left ideals.] Iterating this process, we construct a left ideal $J_{1} \oplus J_{2} \oplus \cdots$, and again by (9.1.2(c)),

$$
R=J_{0} \oplus J_{1} \oplus J_{2} \oplus \cdots .
$$

Therefore 1 is a finite sum of elements $y_{i} \in J_{i}, i=0,1, \ldots, t$. But then

$$
R=J_{0} \oplus J_{1} \oplus \cdots \oplus J_{t}
$$

and it follows that $J_{t+1}$ must be 0 , a contradiction.

## Problems For Section 9.4

Problems 1-5 are the key steps in showing that a ring $R$ is simple if and only if $R$ is Artinian and has no two-sided ideals except 0 and $R$. Thus if a simple ring is defined as one with no nontrivial two-sided ideals, then the addition of the Artinian condition gives our definition of simple ring; in particular, it forces the ring to be semisimple. The result that an Artinian ring with no nontrivial two-sided ideals is isomorphic to a matrix ring over a division ring (Theorem 9.4.7) is sometimes called the Wedderburn-Artin theorem.

In Problems 1-5, "simple" will always mean simple in our sense.

1. By (9.4.5), a simple ring has no nontrivial two-sided ideals. Show that a simple ring must be Artinian.
2. If $R$ is an Artinian ring, show that there exists a simple $R$-module.
3. Let $R$ be an Artinian ring with no nontrivial two-sided ideals. Show that $R$ has a faithful, simple $R$-module.
4. Continuing Problem 3 , if $V$ is a faithful, simple $R$-module, and $D=\operatorname{End}_{R}(V)$, show that $V$ is a finite-dimensional vector space over $D$.
5. Continuing Problem 4, show that $R$ is ring-isomorphic to $\operatorname{End}_{D}(V)$, and therefore to a matrix ring $M_{n}\left(D^{o}\right)$ over a division ring.

In the next section, we will prove that a matrix ring over a division ring is simple; this concludes the proof that $R$ is simple iff $R$ is Artinian with no nontrivial two-sided ideals. (In the "if" part, semisimplicity of $R$ follows from basic properties of matrix rings; see Section 2.2, Problems 2, 3 and 4.)
6. If an $R$-module $M$ is a direct sum $\oplus_{i=1}^{n} M_{i}$ of finitely many simple modules, show that $M$ has a composition series. (Equivalently, by (7.5.12), $M$ is Artinian and Noetherian.)
7. Conversely, if $M$ is semisimple and has a composition series, show that $M$ is a finite direct sum of simple modules. (Equivalently, by Section 9.3, Problems 1 and $2, M$ is finitely generated.)

### 9.5 The Structure of Semisimple Rings

We have now done all the work needed for the fundamental theorem.
9.5.1 Wedderburn Structure Theorem Let $R$ be a semisimple ring.
(1) $R$ is ring-isomorphic to a direct product of simple rings $B_{1}, \ldots, B_{t}$.
(2) There are $t$ isomorphism classes of simple $R$-modules. If $V_{1}, \ldots, V_{t}$ are representatives of these classes, let $D_{i}$ be the division ring $\operatorname{End}_{R}\left(V_{i}\right)$. Then $V_{i}$ is a finite-dimensional vector space over $D_{i}$. If $n_{i}$ is the dimension of this vector space, then there is a ring isomorphism

$$
B_{i} \cong \operatorname{End}_{D_{i}}\left(V_{i}\right) \cong M_{n_{i}}\left(D_{i}^{o}\right)
$$

Consequently, $R$ is isomorphic to the direct product of matrix rings over division rings. Moreover,
(3) $B_{i} V_{j}=0, i \neq j ; B_{i} V_{i}=V_{i}$.

Proof. Assertion (1) follows from (9.3.5), (9.3.9) and (9.3.14). By (9.3.8) and (9.3.12), there are $t$ isomorphism classes of simple $R$-modules. The remaining statements of (2) follow from (9.4.7). The assertions of (3) follow from (9.3.13) and its proof.

Thus a semisimple ring can always be assembled from matrix rings over division rings. We now show that such matrix rings can never combine to produce a ring that is not semisimple.
9.5.2 Theorem The ring $M_{n}(R)$ of all $n$ by $n$ matrices with entries in the division ring $R$ is simple.

Proof. We have done most of the work in the exercises for Section 2.2. Let $C_{k}$ be the set of matrices whose entries are 0 except perhaps in column $k, k=1 \ldots, n$. Then $C_{k}$ is a left ideal of $M_{n}(R)$, and if any nonzero matrix in $C_{k}$ belongs to a left ideal $I$, then $C_{k} \subseteq I$. (Section 2.2, Problems 2,3,4.) Thus each $C_{k}$ is a simple left ideal, and $M_{n}(R)$, the direct sum of $C_{1}, \ldots, C_{n}$, is semisimple.

Now let $I$ be a nonzero simple left ideal. A nonzero matrix in $I$ must have a nonzero entry in some column, say column $k$. Define $f: I \rightarrow C_{k}$ by $f(A)=A_{k}$, the matrix obtained from $A$ by replacing every entry except those in column k by 0 . Then $f$ is an $M_{n}(R)$-module homomorphism, since

$$
f(B A)=(B A)_{k}=B A_{k}=B f(A) .
$$

By construction, $f$ is not identically 0 , so by Schur's lemma, $f$ is an isomorphism. Since the $C_{k}$ are mutually isomorphic, all simple left ideals are isomorphic, proving that $M_{n}(R)$ is simple.

### 9.5.3 Informal Introduction to Group Representations

A major application of semisimple rings and modules occurs in group representation theory, and we will try to indicate the connection. Let $k$ be any field, and let $G$ be a finite group. We form the group algebra $k G$, which is a vector space over $k$ with basis vectors corresponding to the elements of $G$. In general, if $G=\left\{x_{1}, \ldots, x_{m}\right\}$, the elements of $k G$ are of the form $\alpha_{1} x_{1}+\cdots+\alpha_{m} x_{m}$, where the $\alpha_{i}$ belong to $k$. Multiplication in $k G$ is defined in the natural way; we set

$$
\left(\alpha x_{i}\right)\left(\beta x_{j}\right)=\alpha \beta x_{i} x_{j}
$$

and extend by linearity. Then $k G$ is a ring (with identity $1_{k} 1_{G}$ ) that is also a vector space over $k$, and $\alpha(x y)=(\alpha x) y=x(\alpha y), \alpha \in k, x, y \in G$, so $k G$ is indeed an algebra over $k$. [This construction can be carried out with an arbitrary ring $R$ in place of $k$, and with an arbitrary (not necessarily finite) group $G$. The result is the group ring $R G$, a free $R$-module with basis $G$.]

Now let $V$ be an $n$-dimensional vector space over $k$. We want to describe the situation in which " $G$ acts linearly on $V$ ". We are familiar with group action (Section 5.1), but we now add the condition that each $g \in G$ determines a linear transformation $\rho(g)$ on $V$. We will write $\rho(g)(v)$ as simply $g v$ or $g(v)$, so that $g(\alpha v+\beta w)=\alpha g(v)+\beta g(w)$. Thus we can multiply vectors in $V$ by scalars in $G$. Since elements of $k G$ are linear combinations of elements of $G$ with coefficients in $k$, we can multiply vectors in $V$ by scalars in $k G$. To summarize very compactly,
$V$ is a $k G$-module.
Now since $G$ acts on $V,(h g) v=h(g v)$ and $1_{G} v=v, \quad g, h \in G, v \in V$. Thus $\rho(h g)=$ $\rho(h) \rho(g)$, and each $\rho(g)$ is invertible since $\rho(g) \rho\left(g^{-1}\right)=\rho\left(1_{G}\right)=$ the identity on $V$. Therefore $\rho$ is a homomorphism from $G$ to $G L(V)$,
the group of invertible linear transformations on $V$. Multiplication in $G L(V)$ corresponds to composition of functions.

The homomorphism $\rho$ is called a representation of $G$ in $V$,
and $n$, the dimension of $V$, is called the degree of the representation. If we like, we can replace $G L(V)$ by the group of all nonsingular $n$ by $n$ matrices with entries in $k$. In this case, $\rho$ is called a matrix representation.

The above process can be reversed. Given a representation $\rho$, we can define a linear action of $G$ on $V$ by $g v=\rho(g)(v)$, and thereby make $V$ a $k G$-module. Thus representations can be identified with $k G$-modules.
9.5.4 The Regular Representation If $G$ has order $n$, then $k G$ is an $n$-dimensional vector space over $k$ with basis $G$. We take $V$ to be $k G$ itself, with $g v$ the product of $g$ and $v$ in $k G$. As an example, let $G=\left\{e, a, a^{2}\right\}$, a cyclic group of order 3 . $V$ is a 3 -dimensional vector space with basis $e, a, a^{2}$, and the action of $G$ on $V$ is determined by

$$
\begin{gathered}
e e=e, e a=a, e a^{2}=a^{2} \\
a e=a, a a=a^{2}, a a^{2}=e \\
a^{2} e=a^{2}, a^{2} a=e, a^{2} a^{2}=a
\end{gathered}
$$

Thus the matrices $\rho(g)$ associated with the elements $g \in G$ are

$$
[e]=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right],[a]=\left[\begin{array}{lll}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right],\left[a^{2}\right]=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}\right] .
$$

9.5.5 The Role of Semisimplicity Suppose that $\rho$ is a representation of $G$ in $V$. Assume that the basis vectors of $V$ can be decomposed into two subsets $v(A)$ and $v(B)$ such that matrix of every $g \in G$ has the form

$$
[g]=\left[\begin{array}{cc}
A & 0 \\
0 & B
\end{array}\right]
$$

(The elements of $A$ and $B$ will depend on the particular $g$, but the dimensions of $A$ and $B$ do not change.) The corresponding statement about $V$ is that

$$
V=V_{A} \oplus V_{B}
$$

where $V_{A}$ and $V_{B}$ are $k G$-submodules of $V$. We can study the representation by analyzing its behavior on the simpler spaces $V_{A}$ and $V_{B}$. Maschke's theorem, to be proved in the next section, says that under wide conditions on the field $k$, this decomposition process can be continued until we reach subspaces that have no nontrivial $k G$-submodules. In other words, every $k G$-module is semisimple. In particular, $k G$ is a semisimple ring, and the Wedderburn structure theorem can be applied to get basic information about representations.

We will need some properties of projection operators, and it is convenient to take care of this now.
9.5.6 Definitions and Comments A linear transformation $\pi$ on a vector space $V$ [or more generally, a module homomorphism] is called a projection of $V$ (on $\pi(V)$ ) if $\pi$ is idempotent, that is, $\pi^{2}=\pi$. We have already met the natural projection of a direct sum onto a component, but there are other possibilities. For example, let $p$ be the projection of $\mathbb{R}^{2}=\mathbb{R} \oplus \mathbb{R}$ given by $p(x, y)=\left(\frac{x-y}{2}, \frac{-x+y}{2}\right)$. Note that $\pi$ must be the identity on $\pi(V)$, since $\pi(\pi(v))=\pi(v)$.

If we choose subspaces carefully, we can regard any projection as natural.
9.5.7 Proposition If $\pi$ is a projection of $V$, then $V$ is the direct sum of the image of $\pi$ and the kernel of $\pi$.

Proof. Since $v=\pi(v)+(v-\pi(v))$ and $\pi(v-\pi(v))=0, V=\operatorname{im} V+\operatorname{ker} V$. To show that the sum is direct, let $v=\pi(w) \in \operatorname{ker} \pi$. Then $0=\pi(v)=\pi^{2}(w)=\pi(w)=v$, so $\operatorname{im} \pi \cap \operatorname{ker} \pi=0$.
9.5.8 Example For real numbers $x$ and $y$, we have $(x, y)=(x-c y)(1,0)+y(c, 1)$, where $c$ is any fixed real number. Thus $\mathbb{R}^{2}=\mathbb{R}(1,0) \oplus \mathbb{R}(c, 1)$, and if we take $p(x, y)=(x-c y, 0)$, then $p$ is a projection of $\mathbb{R}^{2}$ onto $\mathbb{R}(1,0)$. By varying $c$ we can change the complementary subspace $\mathbb{R}(c, 1)$. Thus we have many distinct projections onto the same subspace $\mathbb{R}(1,0)$.

## Problems For Section 9.5

1. Show that the regular representation is faithful, that is, the homomorphism $\rho$ is injective. 2. Let $G$ be a subgroup of $S_{n}$ and let $V$ be an $n$-dimensional vector space over $k$ with basis $v(1), \ldots, v(n)$. Define the action of $G$ on $V$ by

$$
g(v(i))=v(g(i)), i=1, \ldots, n
$$

Show that the action is legal. ( $V$ is called a permutation module.)
3. Continuing Problem 2, if $n=4$, find the matrix of $g=(1,4,3)$.
4. Here is an example of how a representation can arise in practice. Place an equilateral triangle in the plane $V$, with the vertices at $v_{1}=(1,0), v_{2}=\left(-\frac{1}{2}, \frac{1}{2} \sqrt{3}\right)$ and $v_{3}=\left(-\frac{1}{2},-\frac{1}{2} \sqrt{3}\right)$; note that $v_{1}+v_{2}+v_{3}=0$. Let $G=D_{6}$ be the group of symmetries of the triangle, with $g=$ counterclockwise rotation by 120 degrees and $h=$ reflection about the horizontal axis. Each member of $D_{6}$ is of the form $g^{i} h^{j}, i=0,1,2, j=0,1$, and induces a linear transformation on $V$. Thus we have a representation of $G$ in $V$ (the underlying field $k$ can be taken as $\mathbb{R}$ ).

With $v_{1}$ and $v_{2}$ taken as a basis for $V$, find the matrices $[g]$ and $[h]$ associated with $g$ and $h$. 5. Continue from Problem 4, and switch to the standard basis $e_{1}=v_{1}=(1,0), e_{2}=(0,1)$. Changing the basis produces an equivalent matrix representation. The matrix representing the element $a \in G$ is now of the form

$$
[a]^{\prime}=P^{-1}[a] P
$$

where the similarity matrix $P$ is the same for every $a \in G$ (the key point).
Find the matrix $P$ corresponding to the switch from $\left\{v_{1}, v_{2}\right\}$ to $\left\{e_{1}, e_{2}\right\}$, and the matrices $[g]^{\prime}$ and $[h]^{\prime}$.
6. Consider the dihedral group $D_{8}$, generated by elements $R$ (rotation) and $F$ (reflection). We assign to $R$ the 2 by 2 matrix

$$
A=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]
$$

and to $F$ the 2 by 2 matrix

$$
B=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]
$$

Show that the above assignment determines a matrix representation of $D_{8}$ of degree 2 .
7. Is the representation of Problem 6 faithful?

A very accessible basic text on group representation theory is "Representations and Characters of Groups" by James and Liebeck.

### 9.6 Maschke's Theorem

We can now prove the fundamental theorem on decomposition of representations. It is useful to isolate the key ideas in preliminary lemmas.
9.6.1 Lemma Let $G$ be a finite group, and $k$ a field whose characteristic does not divide $|G|$ (so that division by $|G|$ is legal). Let $V$ be a $k G$-module, and $\psi$ a linear transformation on $V$ as a vector space over $k$. Define $\theta: V \rightarrow V$ by

$$
\theta(v)=\frac{1}{|G|} \sum_{g \in G} g^{-1} \psi g(v)
$$

Then not only is $\theta$ a linear transformation on the vector space $V$, but it is also a $k G$ homomorphism.

Proof. Since $\psi$ is a linear transformation and $G$ acts linearly on $V$ (see (9.5.3)), $\theta$ is linear. Now if $h \in G$, then

$$
\theta(h v)=\frac{1}{|G|} \sum_{g \in G} g^{-1} \psi g(h v)
$$

As $g$ ranges over all of $G$, so does $g h$. Thus we can let $x=g h, g^{-1}=h x^{-1}$, to obtain

$$
\theta(h v)=\frac{1}{|G|} \sum_{x \in G} h x^{-1} \psi(x v)=h \theta(v)
$$

and the result follows.
9.6.2 Lemma In (9.6.1), suppose that $\psi$ is a projection of $V$ on a subspace $W$ that is also a $k G$-submodule of $V$. Then $\theta$ is also a projection of $V$ on $W$.
Proof. If $v \in W$, then $g(v) \in W$ since $W$ is a $k G$-submodule of $V$. Thus $\psi g(v)=g(v)$ since $\psi$ is a projection on $W$. By definition of $\theta$ we have $\theta(v)=v$. To prove that $\theta^{2}=\theta$, note that since $\psi$ maps $V$ into the $k G$-submodule $W$, it follows from the definition of $\theta$ that $\theta$ also maps $V$ into $W$. But $\theta$ is the identity on $W$, so

$$
\theta^{2}(v)=\theta(\theta(v))=\theta(v)
$$

and $\theta$ is a projection. Since $\theta$ maps into $W$ and is the identity on $W, \theta$ is a projection of $V$ on $W$.
9.6.3 Maschke's Theorem Let $G$ be a finite group, and $k$ a field whose characteristic does not divide $|G|$. If $V$ is a $k G$-module, then $V$ is semisimple.

Proof. Let $W$ be a $k G$-submodule of $V$. Ignoring the group algebra for a moment, we can write $V=W \oplus U$ as vector spaces over $k$. Let $\psi$ be the natural projection of $V$ on $W$, and
define $\theta$ as in (9.6.1). By (9.6.1) and (9.6.2), $\theta$ is a $k G$-homomorphism and also a projection of $V$ on $W$. By (9.5.7), $V=\operatorname{im} \theta \oplus \operatorname{ker} \theta=W \oplus \operatorname{ker} \theta$ as $k G$-modules. By (9.1.2), $V$ is semisimple.

We have been examining the decomposition of a semisimple module into a direct sum of simple modules. Suppose we start with an arbitrary module $M$, and ask whether $M$ can be expressed as $M_{1} \oplus M_{2}$, where $M_{1}$ and $M_{2}$ are nonzero submodules. If so, we can try to decompose $M_{1}$ and $M_{2}$, and so on. This process will often terminate in a finite number of steps.
9.6.4 Definition The module $M$ is decomposable if $M=M_{1} \oplus M_{2}$, where $M_{1}$ and $M_{2}$ are nonzero submodules. Otherwise, $M$ is indecomposable.
9.6.5 Proposition Let $M$ be a module with a composition series; equivalently, by (7.5.12), $M$ is Noetherian and Artinian. Then $M$ can be expressed as a finite direct sum $\oplus_{i=1}^{n} M_{i}$ of indecomposable submodules.

Proof. If the decomposition process does not terminate, infinite ascending and descending chains are produced, contradicting the hypothesis.

As the above argument shows, the hypothesis can be weakened to $M$ Noetherian or Artinian. But (9.6.5) is usually stated along with a uniqueness assertion which uses the stronger hypothesis:

If $M$ has a composition series and $M=\oplus_{i=1}^{n} M_{i}=\oplus_{j=1}^{m} N_{j}$, where the $M_{i}$ and $N_{j}$ are indecomposable submodules, then $n=m$ and the $M_{i}$ are, up to isomorphism, just a rearrangement of the $N_{i}$.

The full result (existence plus uniqueness) is most often known as the Krull-Schmidt Theorem. [One or more of the names Remak, Azumaya and Wedderburn are sometimes added.] The uniqueness proof is quite long (see, for example, Jacobson's Basic Algebra II), and we will not need the result.

Returning to semisimple rings, there is an asymmetry in the definition in that a ring is regarded as a left module over itself, so that submodules are left ideals. We can repeat the entire discussion using right ideals, so that we should distinguish between left-semisimple and right-semisimple rings. However, this turns out to be unnecessary.
9.6.6 Theorem A ring $R$ is left-semisimple if and only if it is right-semisimple.

Proof. If $R$ is left-semisimple, then by (9.5.1), $R$ is isomorphic to a direct product of matrix rings over division rings. But a matrix ring over a division ring is right-simple by (9.5.2) with left ideals replaced by right ideals. Therefore $R$ is right-semisimple. The reverse implication is symmetrical.

## Problems For Section 9.6

1. Let $V$ be the permutation module for $G=S_{3}$ (see Section 9.5, Problem 2), with basis $v_{1}, v_{2}, v_{3}$. Give an example of a nontrivial $k G$-submodule of $V$.

In Problems 2-4, we show that Maschke's theorem can fail if the characteristic of $k$ divides the order of $G$. Let $G=\left\{1, a, \ldots, a^{p-1}\right\}$ be a cyclic group of prime order $p$, and let $V$ be a two-dimensional vector space over the field $\mathbb{F}_{p}$, with basis $v_{1}, v_{2}$. Take the matrix of $a$ as

$$
[a]=\left[\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right]
$$

so that

$$
\left[a^{r}\right]=\left[\begin{array}{ll}
1 & r \\
0 & 1
\end{array}\right]
$$

and $\left[a^{p}\right]$ is the identity.
2. Show that $W$, the one-dimensional subspace spanned by $v_{1}$, is a $k G$-submodule of $V$.
3. Continuing Problem 2, show that $W$ is the only one-dimensional $k G$-submodule of $V$.
4. Continuing Problem 3, show that $V$ is not a semisimple $k G$-module.
5. Show that a semisimple module is Noetherian iff it is Artinian.
6. Let $M$ be a decomposable $R$-module, so that $M$ is the direct sum of nonzero submodules $M_{1}$ and $M_{2}$. Show that $\operatorname{End}_{R}(M)$ contains a nontrivial idempotent $e$ (that is, $e^{2}=e$ with $e$ not the zero map and not the identity).
7. Continuing from Problem 6, suppose conversely that $\operatorname{End}_{R}(M)$ contains a nontrivial idempotent $e$. Show that $M$ is decomposable. (Suggestion: use $e$ to construct idempotents $e_{1}$ and $e_{2}$ that are orthogonal, that is, $e_{1} e_{2}=e_{2} e_{1}=0$.)

### 9.7 The Jacobson Radical

There is a very useful device that will allow us to look deeper into the structure of rings.
9.7.1 Definitions and Comments The Jacobson radical $J(R)$ of a ring $R$ is the intersection of all maximal left ideals of $R$. More generally, the Jacobson radical $J(M)=J_{R}(M)$ of an $R$-module $M$ is the intersection of all maximal submodules of $M$. ["Maximal submodule" will always mean "maximal proper submodule".] If $M$ has no maximal submodule, take $J(M)=M$.

If $M$ is finitely generated, then every submodule $N$ of $M$ is contained in a maximal submodule, by Zorn's lemma. [If the union of a chain of proper submodules is $M$, then the union contains all the generators, hence some member of the chain contains all the generators, a contradiction.] Taking $N=0$, we see that $J(M)$ is a proper submodule of $M$. Since $R$ is finitely generated (by $1_{R}$ ), $J(R)$ is always a proper left ideal.

Semisimplicity of $M$ imposes a severe constraint on $J(M)$.
9.7.2 Proposition If $M$ is semisimple, then $J(M)=0$. Thus in a sense, the Jacobson radical is an "obstruction" to semisimplicity.

Proof. Let $N$ be any simple submodule of $M$. By (9.1.2), $M=N \oplus N^{\prime}$ for some submodule $N^{\prime}$. Now $M / N^{\prime} \cong N$, which is simple, so by the correspondence theorem, $N^{\prime}$ is maximal. Thus $J(M) \subseteq N^{\prime}$, and therefore $J(M) \cap N=0$. Since $M$ is a sum of simple modules (see (9.1.2)), $J(M)=J(M) \cap M=0$

Here is another description of the Jacobson radical.
9.7.3 Proposition $J(R)$ is the intersection of all annihilators of simple $R$-modules.

Proof. By Section 9.1, Problem 3, simple modules are isomorphic to $R / I$ for maximal left ideals $I$. If $r$ annihilates all simple $R$-modules, then for every maximal left ideal $I$, $r$ annihilates $R / I$, in particular, $r$ annihilates $1+I$. Thus $r(1+I)=I$, that is, $r \in I$. Consequently, $r \in J(R)$.

Conversely, assume $r \in J(R)$. If $M$ is a simple $R$-module, choose any nonzero element $x \in M$. The map $f_{x}: R \rightarrow M$ given by $f_{x}(s)=s x$ is an epimorphism by simplicity of $M$. The kernel of $f_{x}$ is the annihilator of $x$, denoted by ann $(x)$. By the first isomorphism theorem, $M \cong R / \operatorname{ann}(x)$. By simplicity of $M, \operatorname{ann}(x)$ is a maximal left ideal, so by hypothesis, $r \in \cap_{x \in M} \operatorname{ann}(x)=\operatorname{ann}(M)$. Thus $r$ annihilates all simple $R$-modules.
9.7.4 Corollary $J(R)$ is a two-sided ideal.

Proof. We noted in (4.2.6) that $\operatorname{ann}(M)$ is a two-sided ideal, and the result follows from (9.7.3).

In view of (9.7.4), one might suspect that the Jacobson radical is unchanged if right rather than left ideals are used in the definition. This turns out to be the case.
9.7.5 Definitions and Comments The element $a \in R$ is left quasi-regular (lqr) if $1-a$ has a left inverse, right quasi-regular (rqr) if $1-a$ has a right inverse, and quasi-regular (qr) if $1-a$ is invertible. Note that if $a$ is both lqr and rqr, it is qr, because if $b(1-a)=(1-a) c=1$, then

$$
b=b 1=b(1-a) c=1 c=c .
$$

9.7.6 Lemma Let $I$ be a left ideal of $R$. If every element of $I$ is lqr, then every element of $I$ is qr.

Proof. If $a \in I$, then we have $b(1-a)=1$ for some $b \in R$. Let $c=1-b$, so that $(1-c)(1-a)=1-a-c+c a=1$. Thus $c=c a-a=(c-1) a \in I$. By hypothesis, $c$ is lqr, so $1-c$ has a left inverse. But we know that $(1-c)$ has a right inverse $(1-a)$ [see above], so $c$ is rqr. By (9.7.5), $c$ is qr and $1-c$ is the two-sided inverse of $1-a$.
9.7.7 Proposition The Jacobson radical $J(R)$ is the largest two-sided ideal consisting entirely of quasi-regular elements.

Proof. First, we show that each $a \in J(R)$ is lqr, so by (9.7.6), each $a \in J(R)$ is qr. If $1-a$ has no left inverse, then $R(1-a)$ is a proper left ideal, which is contained in a maximal left ideal $I$ (as in (2.4.2) or (9.7.1)). But then $a \in I$ and $1-a \in I$, and therefore $1 \in I$, a contradiction.

Now we show that every left ideal (hence every two-sided ideal) $I$ consisting entirely of quasi-regular elements is contained in $J(R)$. If $a \in I$ but $a \notin J(R)$, then for some maximal left ideal $L$ we have $a \notin L$. By maximality of $L$, we have $I+L=R$, so $1=b+c$ for some $b \in I, c \in L$. But then $b$ is quasi-regular, so $c=1-b$ has an inverse, and consequently $1 \in L$, a contradiction.
9.7.8 Corollary $J(R)$ is the intersection of all maximal right ideals of $R$.

Proof. We can reproduce the entire discussion beginning with (9.7.1) with left and right ideals interchanged, and reach exactly the same conclusion, namely that the "right" Jacobson radical is the largest two-sided ideal consisting entirely of quasi-regular elements. It follows that the "left" and "right" Jacobson radicals are identical.

We can now use the Jacobson radical to sharpen our understanding of semisimple modules and rings.
9.7.9 Theorem If $M$ is a nonzero $R$-module, the following conditions are equivalent:
(1) $M$ is semisimple and has finite length, that is, has a composition series;
(2) $M$ is Artinian and $J(M)=0$.

Proof. (1) implies (2) by (7.5.12) and (9.7.2), so assume $M$ Artinian with $J(M)=0$. The Artinian condition implies that the collection of all finite intersections of maximal submodules of $M$ has a minimal element $N$. If $S$ is any maximal submodule of $M$, then $N \cap S$ is a finite intersection of maximal submodules, so by minimality of $N, N \cap S=N$, so $N \subseteq S$. Since $J(M)$ is the intersection of all such $S$, the hypothesis that $J(M)=0$ implies that $N=0$. Thus for some positive integer $n$ we have maximal submodules $M_{1}, \ldots, M_{n}$ such that $\cap_{i=1}^{n} M_{i}=0$.

Now $M$ is isomorphic to a submodule of $M^{\prime}=\oplus_{i=1}^{n}\left(M / M_{i}\right)$. To see this, map $x \in M$ to $\left(x+M_{1}, \ldots, x+M_{n}\right)$ and use the first isomorphism theorem. Since $M^{\prime}$ is a finite direct sum of simple modules, it is semisimple and has a composition series. (See Section 9.4, Problem 6) By (9.1.3) and (7.5.7), the same is true for $M$.
9.7.10 Corollary The ring $R$ is semisimple if and only if $R$ is Artinian and $J(R)=0$.

Proof. By (9.7.9), it suffices to show that if $R$ is semisimple, then it has a composition series. But this follows because $R$ is finitely generated, hence is a finite direct sum of simple modules (see Section 9.3, Problem 1).

The Jacobson radical of an Artinian ring has some special properties.
9.7.11 Definitions and Comments An ideal (or left ideal or right ideal) $I$ of the ring $R$ is nil if each element $x \in I$ is nilpotent, that is, $x^{m}=0$ for some positive integer $m ; I$ is nilpotent if $I^{n}=0$ for some positive integer $n$. Every nilpotent ideal is nil, and the converse holds if $R$ is Artinian, as we will prove.
9.7.12 Lemma If $I$ is a nil left ideal of $R$, then $I \subseteq J(R)$.

Proof. If $x \in I$ and $x^{m}=0$, then $x$ is quasi-regular; the inverse of $1-x$ is $1+x+x^{2}+$ $\cdots+x^{m-1}$. The result follows from the proof of (9.7.7).
9.7.13 Proposition If $R$ is Artinian, then $J(R)$ is nilpotent. Thus by (9.7.11) and (9.7.12), $J(R)$ is the largest nilpotent ideal of $R$, and every nil ideal of $R$ is nilpotent.

Proof. Let $J=J(R)$. The sequence $J \supseteq J^{2} \supseteq \cdots$ stabilizes, so for some $n$ we have $J^{n}=J^{n+1}=\cdots$, in particular, $J^{n}=J^{2 n}$. We claim that $J^{n}=0$. If not, then the collection of all left ideals $Q$ of $R$ such that $J^{n} Q \neq 0$ is nonempty (it contains $J^{n}$ ), hence has a minimal element $N$. Choose $x \in N$ such that $J^{n} x \neq 0$. By minimality of $N, J^{n} x=N$. Thus there is an element $c \in J^{n}$ such that $c x=x$, that is, $(1-c) x=0$. But $c \in J^{n} \subseteq J$, so by (9.7.7), $1-c$ is invertible, and consequently $x=0$, a contradiction.

## Problems For Section 9.7

1. Show that an $R$-module is $M$ cyclic if and only if $M$ is isomorphic to $R / I$ for some left ideal $I$, and in this case we can take $I$ to be ann $(M)$, the annihilator of $M$.
2. Show that the Jacobson radical of an $R$-module $M$ is the intersection of all kernels of homomorphisms from $M$ to simple $R$-modules.
3. If $I=J(R)$, show that $J(R / I)=0$.
4. If $f$ is an $R$-module homomorphism from $M$ to $N$, show that $f(J(M)) \subseteq J(N)$.
5. Assume $R$ commutative, so that $J(R)$ is the intersection of all maximal ideals of $R$. If $a \in R$, show that $a \in J(R)$ if and only if $1+a b$ is a unit for every $b \in R$.
6. If $N$ is a submodule of the Jacobson radical of the $R$-module $M$, show that $J(M) / N=$ $J(M / N)$.

### 9.8 Theorems of Hopkins-Levitzki and Nakayama

From Section 7.5, we know that a Noetherian ring need not be Artinian, and an Artinian module need not be Noetherian. But the latter situation can never arise for rings, because of the following result.
9.8.1 Theorem (Hopkins and Levitzki) Let $R$ be an Artinian ring, and $M$ a finitely generated $R$-module. Then $M$ is both Artinian and Noetherian. In particular, with $M=R$, an Artinian ring is Noetherian.

Proof. By (7.5.9), $M$ is Artinian. Let $J$ be the Jacobson radical of $R$. By Section 9.7, Problem 3, the Jacobson radical of $R / J$ is zero, and since $R / J$ is Artinian by (7.5.7), it is semisimple by (9.7.9). Now consider the sequence

$$
M_{0}=M, M_{1}=J M, M_{2}=J^{2} M, \ldots
$$

By (9.7.13), $J$ is nilpotent, so $M_{n}=0$ for some $n$. Since $J M_{i}=M_{i+1}, J$ annihilates $M_{i} / M_{i+1}$, so by Section 4.2, Problem $6, M_{i} / M_{i+1}$ is an $R / J$-module.

We claim that each $M_{i} / M_{i+1}$ has a composition series.
We can assume that $M_{i} / M_{i+1} \neq 0$, otherwise there is nothing to prove. By (9.3.2), $M_{i} / M_{i+1}$ is semisimple, and by (7.5.7), $M_{i} / M_{i+1}$ is Artinian. [Note that submodules of $M_{i} / M_{i+1}$ are the same, whether we use scalars from $R$ or from $R / J$; see Section 4.2, Problem 6.] By Section 9.6, Problem 5, $M_{i} / M_{i+1}$ is Noetherian, hence has a composition series by (7.5.12).

Now intuitively, we can combine the composition series for the $M_{i} / M_{i+1}$ to produce a composition series for $M$, proving that $M$ is Noetherian. Formally, $M_{n-1} \cong M_{n-1} / M_{n}$ has a composition series. Since $M_{n-2} / M_{n-1}$ has a composition series, so does $M_{n-2}$, by (7.5.7). Iterate this process until we reach $M$.

We now proceed to a result that has many applications in both commutative and noncommutative algebra.
9.8.2 Nakayama's Lemma, Version 1 Let $M$ be a finitely generated $R$-module, and $I$ a two-sided ideal of $R$. If $I \subseteq J(R)$ and $I M=M$, then $M=0$.

Proof. Assume $M \neq 0$, and let $x_{1}, \ldots, x_{n}$ generate $M$, where $n$ is as small as possible. (Then $n \geq 1$ and the $x_{i}$ are nonzero.) Since $x_{n} \in M=I M$, we can write $x_{n}=\sum_{i=1}^{m} b_{i} y_{i}$ for some $b_{i} \in I$ and $y_{i} \in M$. But $y_{i}$ can be expressed in terms of the generators as $y_{i}=\sum_{j=1}^{n} a_{i j} x_{j}$ with $a_{i j} \in R$. Thus

$$
x_{n}=\sum_{i, j} b_{i} a_{i j} x_{j}=\sum_{j=1}^{n} c_{j} x_{j}
$$

where $c_{j}=\sum_{i=1}^{m} b_{i} a_{i j}$. Since $I$ is a right ideal, $c_{j} \in I \subseteq J(R)$. (We need $I$ to be a left ideal to make $I M$ a legal submodule of $M$.) The above equation can be written as

$$
\left(1-c_{n}\right) x_{n}=\sum_{j=1}^{n-1} c_{j} x_{j}
$$

and by (9.7.7), $1-c_{n}$ is invertible. If $n>1$, then $x_{n}$ is a linear combination of the other $x_{i}$ 's, contradicting the minimality of $n$. Thus $n=1$, in which case $\left(1-c_{1}\right) x_{1}=0$, so $x_{1}=0$, again a contradiction.

There is another version of Nakayama's lemma, which we prove after a preliminary result.
9.8.3 Lemma Let $N$ be a submodule of the $R$-module $M, I$ a left ideal of $R$. Then $M=N+I M$ if and only if $M / N=I(M / N)$.
Proof. Assume $M=N+I M$, and let $x+N \in M / N$. Then $x=y+z$ for some $y \in N$ and $z \in I M$. Write $z=\sum_{i=1}^{t} a_{i} w_{i}, a_{i} \in I, w_{i} \in M$. It follows that

$$
x+N=a_{1}\left(w_{1}+N\right)+\cdots+a_{t}\left(w_{t}+N\right) \in I(M / N) .
$$

Conversely, assume $M / N=I(M / N)$, and let $x \in M$. Then

$$
x+N=\sum_{i=1}^{t} a_{i}\left(w_{i}+N\right)
$$

with $a_{i} \in I$ and $w_{i} \in M$. Consequently, $x-\sum_{i=1}^{t} a_{i} w_{i} \in N$, so $x \in N+I M$.
9.8.4 Nakayama's Lemma, Version 2 Let $N$ be a submodule of the $R$-module $M$, with $M / N$ finitely generated over $R$. [This will be satisfied if $M$ is finitely generated over $R$.] If $I$ is a two-sided ideal contained in $J(R)$, and $M=N+I M$, then $M=N$.

Proof. By (9.8.3), $I(M / N)=M / N$, so by (9.8.2), $M / N=0$, hence $M=N$.
Here is an application of Nakayama's lemma.
9.8.5 Proposition Let $R$ be a commutative local ring with maximal ideal $J$ (see (8.5.8)). Let $M$ be a finitely generated $R$-module, and let $V=M / J M$. Then:
(i) $V$ is a finite-dimensional vector space over the residue field $k=R / J$.
(ii) If $\left\{x_{1}+J M, \ldots, x_{n}+J M\right\}$ is a basis for $V$ over $k$, then $\left\{x_{1}, \ldots, x_{n}\right\}$ is a minimal set of generators for $M$.
(iii) Any two minimal generating sets for $M$ have the same cardinality.

Proof.
(i) Since $J$ annihilates $M / J M$, it follows from Section 4.2, Problem 6 , that $V$ is a $k$-module, that is, a vector space over $k$. Since $M$ is finitely generated over $R$, $V$ is finite-dimensional over $k$.
(ii) Let $N=\sum_{i=1}^{n} R x_{i}$. Since the $x_{i}+J M$ generate $V=M / J M$, we have $M=N+J M$. By (9.8.4), $M=N$, so the $x_{i}$ generate $M$. If a proper subset of the $x_{i}$ were to generate $M$, then the corresponding subset of the $x_{i}+J M$ would generate $V$, contradicting the assumption that $V$ is $n$-dimensional.
(iii) A generating set $S$ for $M$ with more than $n$ elements determines a spanning set for $V$, which must contain a basis with exactly $n$ elements. By (ii), $S$ cannot be minimal.

## Problems For Section 9.8

1. Let $a$ be a nonzero element of the integral domain $R$. If $\left(a^{t}\right)=\left(a^{t+1}\right)$ for some positive integer $t$, show that $a$ is invertible.
2. Continuing Problem 1, show that every Artinian integral domain is a field.
3. If $R$ is a commutative Artinian ring, show that every prime ideal of $R$ is maximal.
4. Let $R$ be a commutative Artinian ring. If $\mathcal{S}$ is the collection of all finite intersections of maximal ideals of $R$, then $\mathcal{S}$ is not empty, hence contains a minimal element $I=I_{1} \cap I_{2} \cap$ $\cdots \cap I_{n}$, with the $I_{j}$ maximal. Show that if $P$ is any maximal ideal of $R$, then $P$ must be one of the $I_{j}$. Thus $R$ has only finitely many maximal ideals.
5 . An $R$-module is projective if it is a direct summand of a free module. We will study projective modules in detail in Section 10.5. We bring up the subject now in Problems 5 and 6 to illustrate a nice application of Nakayama's lemma.

Let $R$ be a commutative local ring, and let $M$ be a finitely generated projective module over $R$, with a minimal set of generators $\left\{x_{1}, \ldots, x_{n}\right\}$ (see (9.8.5)). We can assume that for some free module $F$ of rank $n$,

$$
F=M \oplus N
$$

To justify this, let $F$ be free with basis $e_{1}, \ldots, e_{n}$, and map $F$ onto $M$ via $e_{i} \rightarrow x_{i}, i=$ $1, \ldots, n$. If the kernel of the mapping is $K$, then we have a short exact sequence

$$
0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0
$$

which splits since $M$ is projective. [This detail will be covered in (10.5.3).]
Let $J$ be the maximal ideal of $R$, and $k=R / J$ the residue field. Show that

$$
F / J F \cong M / J M \oplus N / J N
$$

6. Continue from Problem 5 and show that $N / J N=0$. It then follows from Nakayama's lemma (9.8.2) that $N=0$, and therefore $M=F$. We conclude that a finitely generated projective module over a commutative local ring is free.
7. We showed in (9.6.6) that there is no distinction between a left and a right-semisimple ring. This is not the case for Noetherian (or Artinian) rings.

Let $X$ and $Y$ be noncommuting indeterminates, in other words, $X Y \neq Y X$, and let $\mathbb{Z}<X, Y>$ be the set of all polynomials in $X$ and $Y$ with integer coefficients. [Elements of $\mathbb{Z}$ do commute with the indeterminates.] We impose the relations $Y^{2}=0$ and $Y X=0$ to produce the ring $R$; formally, $R=\mathbb{Z}<X, Y>/\left(Y^{2}, Y X\right)$.

Consider $I=\mathbb{Z}[X] Y$, the set of all polynomials $f(X) Y, f(X) \in \mathbb{Z}[X]$. Then $I$ is a two-sided ideal of $R$. Show that if $I$ is viewed as a right ideal, it is not finitely generated.

Thus $R$ is not right-Noetherian.
8. Viewed as a left $R$-module, $R=\mathbb{Z}[X] \oplus \mathbb{Z}[X] Y$. Show that $R$ is left-Noetherian.
9. Assume the hypothesis of (9.8.5). If $\left\{x_{1}, \ldots, x_{n}\right\}$ is a minimal generating set for $M$, show that $\left\{\bar{x}_{1}, \ldots, \bar{x}_{n}\right\}$, where $\bar{x}_{i}=x_{i}+J M$, is a basis for $M / J M=V$.
10. Continuing Problem 9 , suppose that $\left\{x_{1}, \ldots, x_{n}\right\}$ and $\left\{y_{1}, \ldots, y_{n}\right\}$ are minimal generating sets for $M$, with $y_{i}=\sum_{j} a_{i j} x_{j}, a_{i j} \in R$. If $A$ is the matrix of the $a_{i j}$, show that the determinant of $A$ is a unit in $R$.

