

Chapter Five

More Dimensions

5.1 The Space \mathbf{R}^n

We are now prepared to move on to spaces of dimension greater than three. These spaces are a straightforward generalization of our Euclidean space of three dimensions. Let n be a positive integer. *The n -dimensional Euclidean space \mathbf{R}^n* is simply the set of all ordered n -tuples of real numbers $\mathbf{x} = (x_1, x_2, \dots, x_n)$. Thus \mathbf{R}^1 is simply the real numbers, \mathbf{R}^2 is the plane, and \mathbf{R}^3 is Euclidean three-space. These ordered n -tuples are called *points*, or *vectors*. This definition does not contradict our previous definition of a vector in case $n = 3$ in that we identified each vector with an ordered triple (x_1, x_2, x_3) and spoke of the triple as being a vector.

We now define various arithmetic operations on \mathbf{R}^n in the obvious way. If we have vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ in \mathbf{R}^n , the sum $\mathbf{x} + \mathbf{y}$ is defined by

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

and multiplication of the vector \mathbf{x} by a scalar a is defined by

$$a\mathbf{x} = (ax_1, ax_2, \dots, ax_n).$$

It is easy to verify that $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$ and $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

Again we see that these definitions are entirely consistent with what we have done in dimensions 1, 2, and 3—there is nothing to unlearn. Continuing, we define the *length*, or *norm* of a vector \mathbf{x} in the obvious manner

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

The *scalar product* of \mathbf{x} and \mathbf{y} is

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n = \sum_{i=1}^n x_iy_i.$$

It is again easy to verify the nice properties:

$$|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x} \geq 0,$$

$$|a\mathbf{x}| = |a||\mathbf{x}|,$$

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x},$$

$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}, \text{ and}$$

$$(a\mathbf{x}) \cdot \mathbf{y} = a(\mathbf{x} \cdot \mathbf{y}).$$

The geometric language of the three dimensional setting is retained in higher dimensions; thus we speak of the “length” of an n -tuple of numbers. In fact, we also speak of $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ as the *distance* between \mathbf{x} and \mathbf{y} . We can, of course, no longer rely on our vast knowledge of Euclidean geometry in our reasoning about \mathbf{R}^n when $n > 3$.

Thus for $n = 3$, the fact that $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for any vectors \mathbf{x} and \mathbf{y} was a simple consequence of the fact that the sum of the lengths of two sides of a triangle is at least as big as the length of the third side. This inequality remains true in higher dimensions, and, in fact, is called the *triangle inequality*, but requires an essentially algebraic proof. Let's see if we can prove it.

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$. Then if a is a scalar, we have

$$|a\mathbf{x} + \mathbf{y}|^2 = (a\mathbf{x} + \mathbf{y}) \cdot (a\mathbf{x} + \mathbf{y}) \geq 0.$$

Thus,

$$(a\mathbf{x} + \mathbf{y}) \cdot (a\mathbf{x} + \mathbf{y}) = a^2\mathbf{x} \cdot \mathbf{x} + 2a\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \geq 0.$$

This is a quadratic function in a and is never negative; it must therefore be true that

$$4(\mathbf{x} \cdot \mathbf{y})^2 - 4(\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y}) \leq 0, \text{ or}$$

$$|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|.$$

This last inequality is the celebrated *Cauchy-Schwarz-Buniakowsky inequality*. It is exactly the ingredient we need to prove the triangle inequality.

$$|\mathbf{x} + \mathbf{y}|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}.$$

Applying the **C-S-B** inequality, we have

$$|\mathbf{x} + \mathbf{y}|^2 \leq |\mathbf{x}|^2 + 2|\mathbf{x}||\mathbf{y}| + |\mathbf{y}|^2 = (|\mathbf{x}| + |\mathbf{y}|)^2, \text{ or}$$

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|.$$

Corresponding to the coordinate vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} , the coordinate vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are defined in \mathbf{R}^n by

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, 0, \dots, 0) \\ \mathbf{e}_2 &= (0, 1, 0, \dots, 0) \\ \mathbf{e}_3 &= (0, 0, 1, \dots, 0), \\ &\vdots \\ \mathbf{e}_n &= (0, 0, 0, \dots, 0, 1) \end{aligned}$$

Thus each vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ may be written in terms of these coordinate vectors:

$$\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i.$$

Exercises

- Let \mathbf{x} and \mathbf{y} be two vectors in \mathbf{R}^n . Prove that $|\mathbf{x} + \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$. (Adopting more geometric language from three space, we say that \mathbf{x} and \mathbf{y} are *perpendicular* or *orthogonal* if $\mathbf{x} \cdot \mathbf{y} = 0$.)
- Let \mathbf{x} and \mathbf{y} be two vectors in \mathbf{R}^n . Prove
 - $|\mathbf{x} + \mathbf{y}|^2 - |\mathbf{x} - \mathbf{y}|^2 = 4\mathbf{x} \cdot \mathbf{y}$.
 - $|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2[|\mathbf{x}|^2 + |\mathbf{y}|^2]$.

3. Let \mathbf{x} and \mathbf{y} be two vectors in \mathbf{R}^n . Prove that $||\mathbf{x}| - |\mathbf{y}|| \leq |\mathbf{x} + \mathbf{y}|$.

4. Let $P \subset \mathbf{R}^4$ be the set of all vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ such that

$$3x_1 + 5x_2 - 2x_3 + x_4 = 15.$$

Find vectors \mathbf{n} and \mathbf{a} such that $P = \{\mathbf{x} \in \mathbf{R}^4 : \mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0\}$.

5. Let \mathbf{n} and \mathbf{a} be vectors in \mathbf{R}^n , and let $P = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0\}$.

a) Find an equation in x_1, x_2, \dots, x_n such that $\mathbf{x} = (x_1, x_2, \dots, x_n) \in P$ if and only if the coordinates of \mathbf{x} satisfy the equation.

b) Describe the set P in case $n = 3$. Describe it in case $n = 2$.

[The set P is called a *hyperplane through \mathbf{a}* .]

5.2 Functions

We now consider functions $F: \mathbf{R}^n \rightarrow \mathbf{R}^p$. Note that when $n = p = 1$, we have the usual grammar school calculus functions, and when $n = 1$ and $p = 2$ or 3 , we have the vector valued functions of the previous chapter. Note also that except for very special circumstances, graphs of functions will not play a big role in our understanding. The set of points $(\mathbf{x}, F(\mathbf{x}))$ resides in \mathbf{R}^{n+p} since $\mathbf{x} \in \mathbf{R}^n$ and $F(\mathbf{x}) \in \mathbf{R}^p$; this is difficult to “see” unless $n + p = 3$.

We begin with a very special kind of functions, the so-called linear functions. A function $F: \mathbf{R}^n \rightarrow \mathbf{R}^p$ is said to be a *linear* function if

$$\text{i) } F(\mathbf{x} + \mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbf{R}^n, \text{ and}$$

$$\text{ii) } F(a\mathbf{x}) = aF(\mathbf{x}) \text{ for all scalars } a \text{ and } \mathbf{x} \in \mathbf{R}^n.$$

Example

Let $n = p = 1$, and define F by $F(x) = 3x$. Then

$$F(x + y) = 3(x + y) = 3x + 3y = F(x) + F(y) \text{ and}$$

$$F(ax) = 3(ax) = a3x = aF(x).$$

This F is a linear function.

Another Example

Let $F: \mathbf{R} \rightarrow \mathbf{R}^3$ be defined by $F(t) = ti + 2tj - 7tk = (t, 2t, -7t)$. Then

$$\begin{aligned} F(t+s) &= (t+s)i + 2(t+s)j - 7(t+s)k \\ &= [ti + 2tj - 7tk] + [si + 2sj - 7sk] \\ &= F(t) + F(s) \end{aligned}$$

Also,

$$\begin{aligned} F(at) &= ati + 2atj - 7atk \\ &= a[ti + 2tj - 7tk] = aF(t) \end{aligned}$$

We see yet another linear function.

One More Example

Let $F: \mathbf{R}^3 \rightarrow \mathbf{R}^4$ be defined by

$$F((x_1, x_2, x_3)) = (2x_1 - x_2 + 3x_3, x_1 + 4x_2 - 5x_3, -x_1 + 2x_2 + x_3, x_1 + x_3).$$

It is easy to verify that F is indeed a linear function.

A **translation** is a function $T: \mathbf{R}^p \rightarrow \mathbf{R}^p$ such that $T(\mathbf{x}) = \mathbf{a} + \mathbf{x}$, where \mathbf{a} is a fixed vector in \mathbf{R}^p . A function that is the composition of a linear function followed by a translation is called an **affine** function. An affine function F thus has the form $F(\mathbf{x}) = \mathbf{a} + L(\mathbf{x})$, where L is a linear function.

Example

Let $F: \mathbf{R} \rightarrow \mathbf{R}^3$ be defined by $F(t) = (2+t, 4t-3, t)$. Then F is affine. Let $\mathbf{a} = (2, 4, 0)$ and $L(t) = (t, 4t, t)$. Clearly $F(t) = \mathbf{a} + L(t)$.

Exercises

6. Which of the following functions are linear? Explain your answers.

a) $f(x) = -7x$

b) $g(x) = 2x - 5$

c) $F(x_1, x_2) = (2x_1 + x_2, x_1 - x_2, 3x_1, 5x_1 - 2x_2, x_1)$

d) $G(x_1, x_2, x_3) = x_1x_2 + x_3$

e) $F(t) = (2t, t, 0, -2t)$

f) $h(x_1, x_2, x_3, x_4) = (1, 0, 0)$

g) $f(x) = \sin x$

7. a) Describe the graph of a linear function from \mathbf{R} to \mathbf{R} .
b) Describe the graph of an affine function from \mathbf{R} to \mathbf{R} .